1. (6 points) Evaluate the following limits, showing your work and/or explaining your answers.

(a) $\lim_{x\to 2} \frac{\left(\frac{1}{x^2}-\frac{1}{4}\right)}{\left(\frac{1}{x}-\frac{1}{2}\right)} \frac{4x^2}{4x^2} = \lim_{x\to 2} \frac{4-x^2}{4x-2x^2}$

$= \lim_{x\to 2} \frac{(2-x)(2+x)}{2x(2-x)} = \frac{2+(2)}{2(2)} = \boxed{1}$

(b) $\lim_{x\to\infty} \frac{\sin(x)}{x} = \boxed{0}$

$-1 \le \sin(x) \le 1$

$\Rightarrow -\frac{1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$

$\lim_{x\to\infty} -\frac{1}{x} = 0$ AND

$\lim_{x\to\infty} \frac{1}{x} = 0$

squeeze thm

2. (8 points) Compute the derivatives of the following functions. You need not simplify your answers.

c) $f(x) = \cos^2(\tan(x))$

$f'(x) = 2\cos(\tan(x))(-\sin(\tan(x)))\sec^2(x)$

$= \boxed{-2\cos(\tan(x))\sin(\tan(x))\sec^2(x)}$

d) $f(x) = \ln(2x^{\sin x}) = \ln(2) + \ln\left(x^{\sin(x)}\right)$

$\Rightarrow f(x) = \ln(2) + \sin(x)\ln(x)$

$f'(x) = 0 + \cos(x)\ln(x) + \sin(x)\frac{1}{x}$

$= \boxed{\cos(x)\ln(x) + \frac{\sin(x)}{x}}$

2. (12 total points) Find the following limits. In each case your answer should be either a number, $+\infty$, $-\infty$ or DNE. Please show your work.

(a) (4 points) $\lim_{t\longrightarrow 2^-} \frac{t^2-4}{|t-2|}$ $= \lim_{t\to 2^-} \frac{(t-2)(t+2)}{|t-2|} = \lim_{t\to 2^-} \frac{\cancel{(t-2)}(t+2)}{-\cancel{(t-2)}}$

$\underbrace{\text{FOR } t<2,\ |t-2| = -(t-2)}$ $\Rightarrow$

$= -(2+2)$

$= \boxed{-4}$

OR NOTICE THAT $\lim_{t\to 2^-} \frac{t-2}{|t-2|} = -1$

(b) (4 points) $\lim_{x\longrightarrow\infty}\left(x-\sqrt{x^2-10x}\right)$ $\frac{x+\sqrt{x^2-10x}}{x+\sqrt{x^2-10x}}$ ← TURN INTO A FRACTION

$= \lim_{x\to\infty} \frac{x^2-(x^2-10x)}{x+\sqrt{x^2-10x}}$

FOR $x>0$, $x=\sqrt{x^2}$

$= \lim_{x\to\infty} \frac{10x}{x+\sqrt{x^2-10x}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{10}{1+\sqrt{\frac{x^2-10x}{x^2}}}$

$= \lim_{x\to\infty} \frac{10}{1+\sqrt{1-10/x}} = \frac{10}{1+\sqrt{1-0}} = \frac{10}{2} = \boxed{5}$

(c) (4 points) $\lim_{x\longrightarrow\infty} \frac{2x^2+3x\ln x+2^{-x}}{5x^2+9x\ln x+\pi\cdot 2^{-x}}$ $\frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

CAN USE L'HOPITAL'S RULE

$\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$

$= \lim_{x\to\infty} \frac{2+\frac{3\ln(x)}{x}+\frac{1}{2^x}\cancel{\frac{1}{x^2}}^{0}}{5+\frac{9\ln(x)}{x}+\frac{\pi}{\cancel{2^x}}\cancel{\frac{1}{x^2}}^{0}}$

$= \frac{2+0+0}{5+0+0} = \boxed{2/5}$

WIN 2014

6. (18 Points)
Consider the function

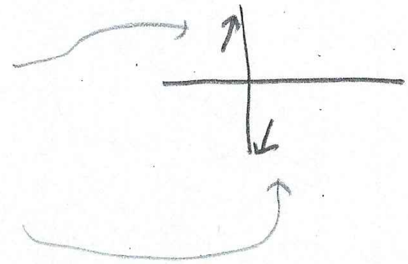
$$f(x) = \frac{x^2 - 3}{x^3} = \frac{1}{x} - \frac{3}{x^3} = x^{-1} - 3x^{-3}$$

(a) Find all vertical **asymptotes** or state that there are none. Justify your answer with limit computations.

$\boxed{x = 0}$

$$\lim_{x \to 0^-} \frac{x^2 - 3}{x^3} \overset{\frac{-3}{0^-}}{=} +\infty$$

$$\lim_{x \to 0^+} \frac{x^2 - 3}{x^3} \overset{\frac{-3}{0^+}}{=} -\infty$$



(b) Find all horizontal **asymptotes** or state that there are none. Justify your answer with limit computations.

$$\lim_{x \to \pm\infty} \frac{(x^2 - 3)\ \frac{1}{x^3}}{(x^3)\ \frac{1}{x^3}} = \lim_{x \to \pm\infty} \frac{1}{x} - \frac{3}{x^3} = 0$$

$\boxed{y = 0}$

(c) Find all critical points (numbers) of f and determine whether each corresponds to a local minimum, local maximum, or neither.

$$f'(x) = -x^{-2} + 9x^{-4} = \left(-\frac{1}{x^2} + \frac{9}{x^4} \overset{?}{=} 0\right) x^4$$

$$\Rightarrow -x^2 + 9 = 0$$

$$\Rightarrow x = \pm 3$$

f	$\searrow$		$\nearrow$		$\nearrow$		$\searrow$
f'	$-$	-3	$+$	0	$+$	3	$-$

$$\boxed{\begin{array}{ll} x = -3, & \text{local min} \\ x = 3, & \text{local max} \end{array}}$$

WIN 2014

(d) Find all inflection points of f, and list the intervals on which f is concave up.

$$f''(x) = 2x^{-3} - 36x^{-5} = \left(\frac{2}{x^3} - \frac{36}{x^5} \overset{?}{=} 0\right) x^5$$

$$\Rightarrow \quad 2x^2 - 36 = 0$$

$$\Rightarrow \quad x^2 = 18 \Rightarrow x = \pm\sqrt{18} = \pm 3\sqrt{2}$$

f: ∩ , ∪ , ∩ , ∪

f'': $-$ $\;-\sqrt{18}\;$ $+$ $\;0\;$ $-$ $\;\sqrt{18}\;$ $+$

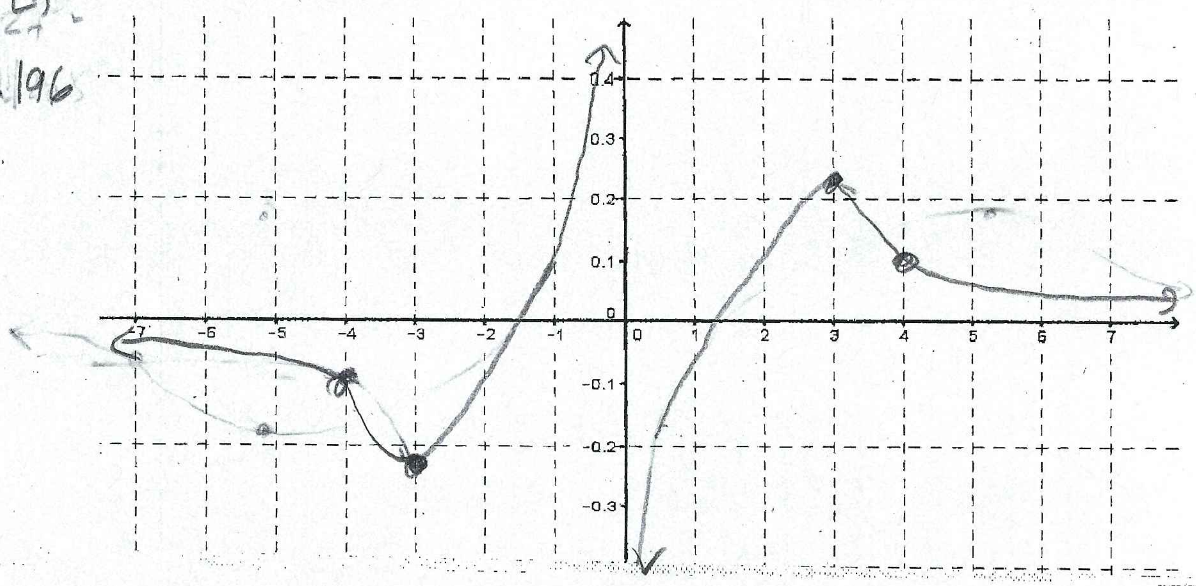
INFLECTIONS PTS: $\boxed{\begin{array}{l} x = -\sqrt{18} \\ x = \sqrt{18} \end{array}} \approx 4.2426$

CONCAVE UP $(-\sqrt{18}, 0)$ and $(\sqrt{18}, \infty)$

(e) Carefully and clearly sketch the graph of $y = f(x)$. Include BOTH coordinates of all points on the graph that correspond to critical points and inflection points.

$f(3) = \frac{9-3}{27} = 6/27 = 2/9 \approx 0.\overline{2}$

$f(\sqrt{18}) \approx 0.196$



8. (16 total points) Let $f(x)$ be the function

$$f(x) = (2x+5)e^{(-x/2)}$$

(a) (2 points) Give the (x,y)-coordinates of all x-intercepts and y-intercepts of $y = f(x)$.

$\boxed{(0,5) \quad (-\frac{5}{2}, 0)}$

$x=0 \Rightarrow y = f(0) = 5$

$y=0 \Rightarrow (2x+5)e^{-x/2} \stackrel{?}{=} 0$

$\Rightarrow x = -5/2$

(b) (2 points) Find the following limits.

i. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x+5}{e^{(x/2)}} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{2}{\frac{1}{2}e^{x/2}} = \boxed{0}$

ii. $\lim_{x\to-\infty} f(x) = \boxed{-\infty}$

(c) (3 points) Find all intervals over which $f(x)$ is increasing.

$f'(x) = 2e^{-x/2} - \frac{1}{2}(2x+5)e^{-x/2} \stackrel{?}{=} 0$

$e^{-x/2}(-\frac{1}{2}-x) = e^{-x/2}\left[2 - \frac{1}{2}(2x+5)\right] \stackrel{?}{=} 0$

$\Rightarrow 2 - x - 5/2 = 0 \Rightarrow x = -1/2$

f	$\nearrow$		$\searrow$
f'	$+$	$-1/2$	$-$

$\boxed{(-\infty, -1/2)}$

8. (continued) Recall that the function is $f(x) = (2x+5)e^{(-x/2)}$

(d) (3 points) Find all intervals over which $f(x)$ is concave down.

$f'(x) = -\frac{1}{2} e^{-x/2}(1+2x)$

$f''(x) = \frac{1}{4} e^{-x/2}(1+2x) - e^{-x/2} \stackrel{?}{=} 0 \longrightarrow e^{-x/2}\left(\frac{1}{2}x - \frac{3}{4}\right)$

$\Rightarrow e^{-x/2}\left(\frac{1}{4} + \frac{1}{2}x - 1\right) = 0$

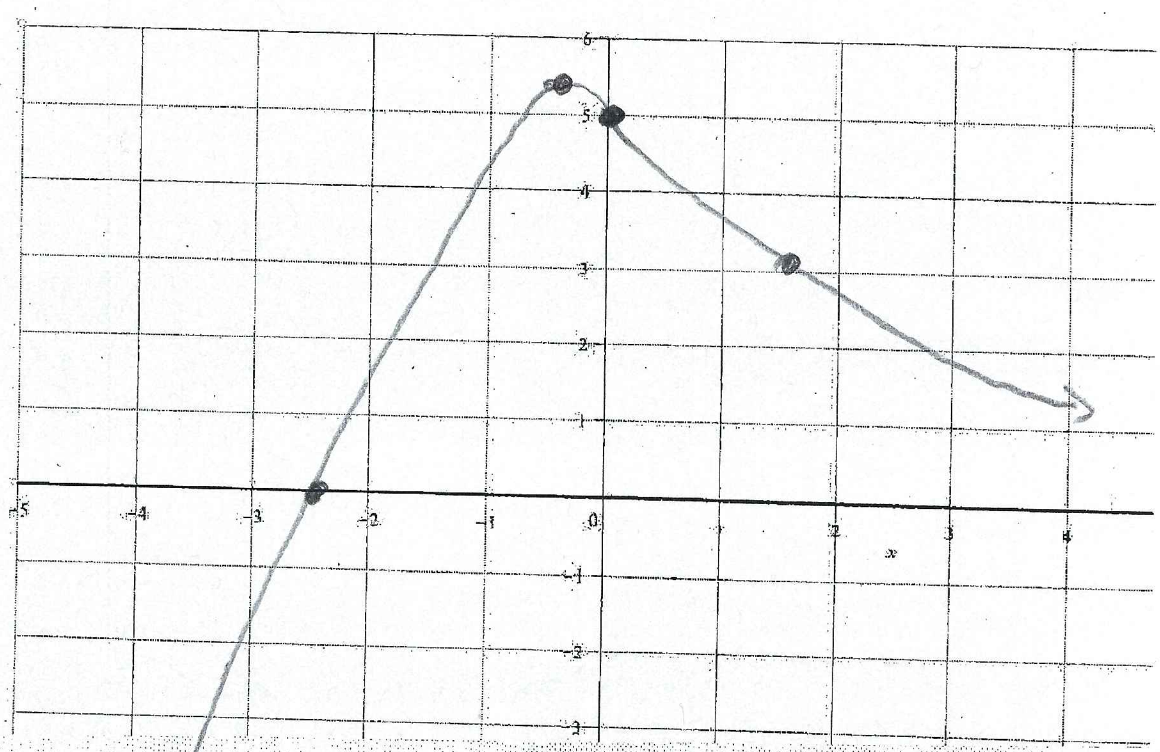
$\Rightarrow \frac{1}{2}x = \frac{3}{4} \Rightarrow x = \frac{3}{2}$

$\boxed{(-\infty, \frac{3}{2})}$

f	$\cap$		$\cup$
f''	$-$	$\frac{3}{2}$	$+$

(e) (6 points) Sketch the graph of $y = f(x)$ using the grid below. Clearly label the (x, y) coordinates of all critical points and all points of inflection.

$f(-\frac{1}{2}) \approx$



5. (12 total points) For time $t > 0$ seconds, a particle moves according to the parametric equations

$$x(t) = 3t^2 - t - 9, \quad y(t) = 9t^3 - 8\ln t$$

(a) (6 points) Find the time t at which the path of the particle in the xy-plane has a horizontal tangent.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \overset{?}{=} 0 \iff dy/dt = 0$$

$$\left(27t^2 - \frac{8}{t} \overset{?}{=} 0\right) t$$

$$27t^3 - 8 = 0$$

$$t^3 = 8/27$$

$$\boxed{t = 2/3}$$

(b) (6 points) At time $t = 1$ second, the particle departs from the parametric curve and continues along the tangent line to the curve at that point. The particle travels along this tangent line at a constant speed, preserving the horizontal and vertical velocities from the moment it left the parametric curve. How many seconds after leaving the curve will the particle cross the y-axis?

$$x(1) = 3(1)^2 - (1) - 9 = 3 - 10 = -7$$

$$y(1) = 9(1)^3 - 8\ln(1) = 9$$

$$\frac{dx}{dt} = 6t - 1, \quad \frac{dx}{dt}(1) = 6(1) - 1 = 5$$

$$\frac{dy}{dt} = 27t^2 - \frac{8}{t}, \quad \frac{dy}{dt}(1) = 27 - 8 = 19$$

$$x = x_0 + v_x t = -7 + 5t$$

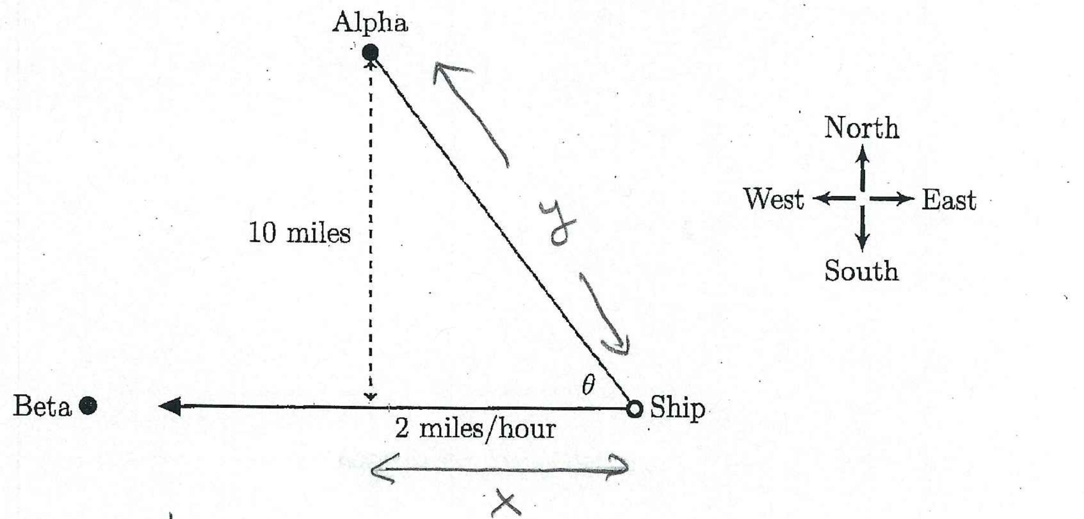
$$y = y_0 + v_y t = 19 + 19t$$

CROSSES y-AXIS $\Rightarrow x = 0 \Rightarrow -7 + 5t \overset{?}{=} 0$

$$\boxed{t = 7/5 = 1.4 \text{ sec}}$$

5. (10 Points) Island Alpha is ten miles north and some distance east of Island Beta. A ship is sailing west towards Island Beta at a speed of 2 miles per hour. The angle θ is measured between the two lines of sight from the ship to each island.

(a) When the ship is 2 miles east (and 10 miles south) of Island Alpha, at what rate is θ changing?



GIVEN: $\frac{dx}{dt} = 2$ WANT: $\frac{d\theta}{dt}$ WHEN $x = 2$

$$\tan\theta = \frac{x}{10} \Rightarrow \theta = \tan^{-1}\left(\frac{x}{10}\right)$$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{x}{10}\right)^2} \cdot \frac{1}{10}\frac{dx}{dt}$$

$$= \frac{1}{1+\left(\frac{2}{10}\right)^2} \frac{1}{10} \cdot (2)$$

$$= \frac{1}{1+1/25} \frac{1}{5} = \frac{1}{5+1/5} = \frac{5}{26}$$

$$\boxed{\frac{d\theta}{dt} = \frac{5}{26} \frac{\text{rad}}{\text{hr}}}$$

(b) At that time, is the angle θ increasing or decreasing?

increasing

3. (14 points) The lateral surface area of a cone of radius r and height h (that is, the surface area excluding the base) is:

$$A = \pi r \sqrt{r^2 + h^2}$$

(a) Find a formula (in terms of r and h) for $\frac{dr}{dh}$ for a cone with lateral surface area $A = 15\pi$ cm^2.

$15\pi = \pi r \sqrt{r^2+h^2}$

$\Rightarrow 15 = r\sqrt{r^2+h^2}$

$\Rightarrow \left(0 = \frac{dr}{dh}\sqrt{r^2+h^2} + r\frac{(2r\frac{dr}{dh}+2h)}{2\sqrt{r^2+h^2}} \right) \quad 2\sqrt{r^2+h^2}$

$\Rightarrow 0 = 2\frac{dr}{dh}(r^2+h^2) + 2r^2\frac{dr}{dh} + 2rh$

$\Rightarrow -2rh = (2r^2+2h^2+2r^2)\frac{dr}{dh}$

$\Rightarrow \boxed{\frac{dr}{dh} = \frac{-2rh}{4r^2+2h^2}}$

(b) Evaluate the derivative of part (a) when $r = 3$ cm and $h = 4$ cm.

$\left.\frac{dr}{dh}\right|_{(r,h)=(3,4)} = \frac{-2(3)(4)}{4(3)^2+2(4)^2} = \frac{-24}{36+32}$

$= -\frac{24}{68} = \boxed{-\frac{6}{17}}$

(c) Suppose that the height of the cone decreases 0.1 cm (from 4.0 to 3.9 cm). Use differentials to approximate how much the radius must increase in order keep the lateral surface area of the cone constant.

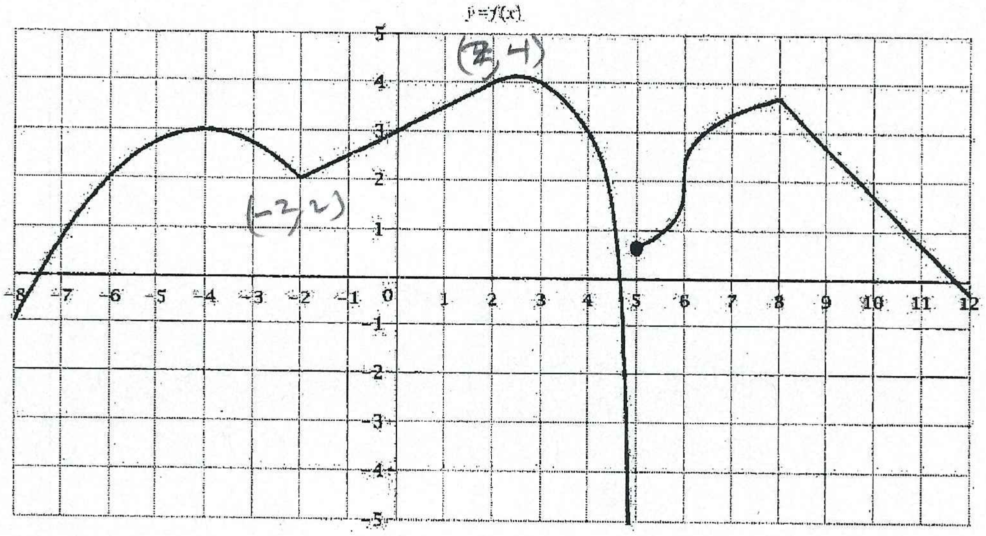
$r = -\frac{6}{17}(h-3)+4$

2.9

$r \approx 4.03529$ cm

$\boxed{\text{Increase about } 0.03529 \text{ cm}}$

3. (12 total points) The following is the graph of the function $f(x)$ with domain $-8 \le x \le 12$. The vertical line $x = 5$ is an asymptote. Answer the following questions based on the graph. You do not need to justify your answers on this problem.



(a) (2 points) List all intervals where the derivative $f'(x)$ is increasing.

$f'(x)$ increasing $\Leftrightarrow f''$ positive $\Leftrightarrow f$ concave up

$\boxed{(5,6)}$

(b) (2 points) $f'(-1) =$

$f'(-1) =$ "slope at $x=-1$" $= \frac{(4-2)}{2--2} = \frac{2}{4} = \boxed{1/2}$

(c) (2 points) $\boxed{f''(0) = 0}$

$f(x) = mx + b$ from $x=-2$ to $x=2$

$f' = m \qquad f'' = 0$

(d) (2 points) $\boxed{\lim_{x\to -2} f(x) = 2}$

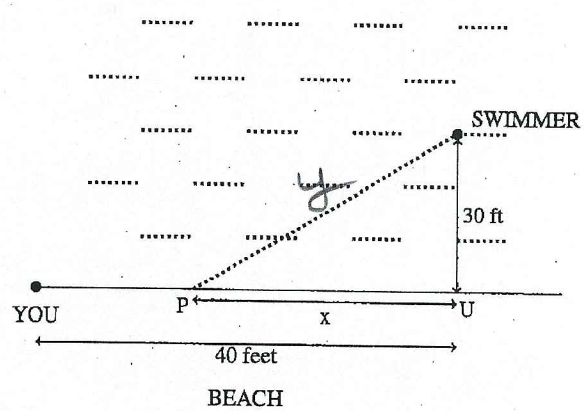
(e) (2 points) $\lim_{x\to -4} \frac{f(x)-3}{x+4} \overset{0/0}{=} \lim_{x\to -4} \frac{f'(x)}{1} = f'(-4) =$ "slope at $x=-4$"

$= \boxed{0}$

4. (12 points) You are standing on the beach when you notice a swimmer in distress. He is 30 feet from shore. At the point on the shore nearest the swimmer there is a big beach umbrella (marked U in the picture below). You are 40 feet down the beach from the umbrella. You want to run along the beach to a point P, then jump into the water and swim in a straight line towards him. Your running speed is 10 feet/sec and your swimming speed is 5 feet/sec.

What should the distance x between P and the umbrella U be if you want to minimize the time it takes you to reach the swimmer?

Verify that your answer is a minumum.



MINIMIZE TIME!

$$TIME = \frac{RUN\ DIST}{10} + \frac{SWIM\ DIST}{5}$$

$$f(x) = \frac{(40-x)}{10} + \frac{\sqrt{900+x^2}}{5}$$

$x^2 + 30^2 = y^2$

$y = \sqrt{900+x^2}$

$$f'(x) = -\frac{1}{10} + \frac{1}{5}\frac{\cancel{2}x}{\cancel{2}\sqrt{900+x^2}} \stackrel{?}{=} 0$$

$$\frac{1}{5}\frac{x}{\sqrt{900-x^2}} = \frac{1}{10}$$

$$\Rightarrow \quad x = \frac{1}{2}\sqrt{900+x^2}$$

$$x^2 = \frac{1}{4}(900+x^2) \Rightarrow 4x^2 = 900+x^2$$

$$\Rightarrow 3x^2 = 900$$

$$x^2 = 300$$

$$\boxed{x = \sqrt{300} = 10\sqrt{3} \approx 17.3205}$$

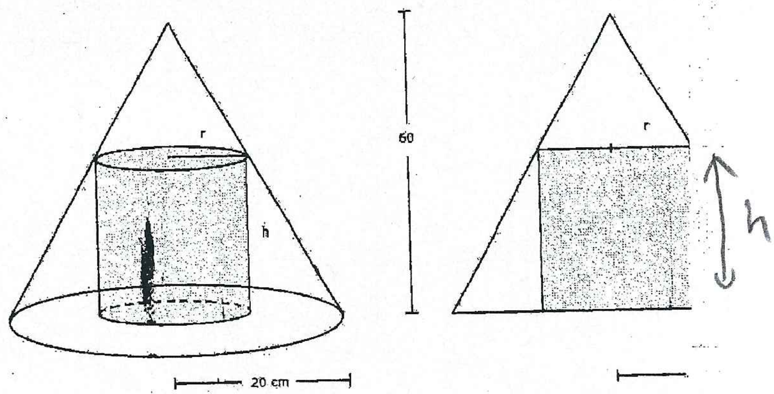
$f(0) = 4 + 6 = 10$ sec

$f(\sqrt{300}) \approx 9.196524$ sec

$f(40) = 0 + 10$ sec

ABS. MIN

7. (14 points) Find the height h and radius r of the cylinder of maximum volume that can be inscribed in a cone of radius 20 centimeters and height 60 centimeters. Make sure you justify why the cylinder with the dimensions you give has maximum possible volume.

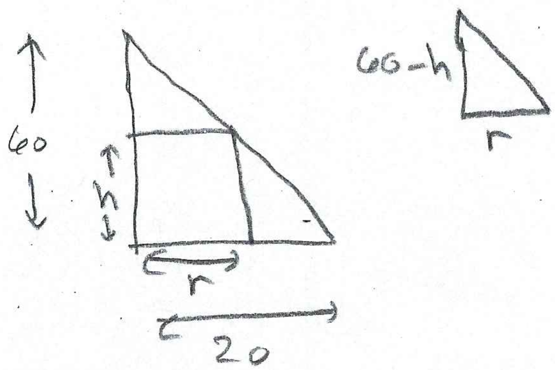


Cylinder inscribed in a cone

picture of cross-sectio

MAXIMIZE VOLUME!

VOLUME $= \pi r^2 h$



$$\frac{r}{20} = \frac{60-h}{60} \Rightarrow 3r = 60 - h$$

$$\Rightarrow h = 60 - 3r$$

$$f(r) = \pi r^2 (60 - 3r)$$

$$= 60\pi r^2 - 3\pi r^3$$

$$f'(r) = 120\pi r - 9\pi r^2 \overset{?}{=} 0$$

$$\pi r (120 - 9r) \overset{!}{=} 0$$

$$r = 0 \quad \text{OR} \quad r = \frac{120}{9} = \frac{40}{3} \approx 13.\overline{3}$$

$$f''(r) = 120\pi - 18\pi r < 0 \quad \text{concave down}$$

MAX!

$$\boxed{r = \frac{40}{3}, \quad h = 60 - 3\left(\frac{40}{3}\right) = 20}$$